Topic :- Hyperbolic Functions [B.Sc. Part I Trigonometry] Date 17/04/2020

Introduction :-

Usual Notations

* $f(\theta) = \cosh\theta$
  $= \sinh\theta$ } are known as basic hyperbolic functions

* $f(\theta) = \tanh\theta,$
  $= \operatorname{sech}\theta$
  $= \operatorname{cosech}\theta$
  $= \coth\theta$ } are derived hyperbolic functions. these are derived from Basic hyperbolic functions.

Here θ → may be real
→ or be complex

* Definition → If θ be real or complex we define as

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

and $$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

Since hyperbolic functions are analogs of circular functions

We can obtain

$$\tanh\theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}, \quad \coth\theta = \frac{e^{\theta} + e^{-\theta}}{e^{\theta} - e^{-\theta}}$$

$$\operatorname{sech}\theta = \frac{2}{e^{\theta} + e^{-\theta}} \qquad \operatorname{cosech}\theta = \frac{2}{e^{\theta} - e^{-\theta}}$$

* The values of $\sinh\theta$, $\cosh\theta$, $\tanh\theta$

When $\theta = -\infty$ $\sinh\theta = -\infty$
$\cosh\theta = \infty$
$\tanh\theta = -1$

$\theta = 0$ $\sinh\theta = 0$
$\cosh\theta = 1$
$\tanh\theta = 0$

$\theta = \infty$ $\sinh\theta = \infty$
$\cosh\theta = \infty$
$\tanh\theta = 1$

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Expansion of Basic hyperbolic function: →

<u>Requirement</u> → $e^{\theta} = 1 + \theta + \frac{\theta^2}{\lfloor 2} + \frac{\theta^3}{\lfloor 3} + \frac{\theta^4}{\lfloor 4} + \cdots$

hence $e^{-\theta} = 1 - \theta + \frac{\theta^2}{\lfloor 2} - \frac{\theta^3}{\lfloor 3} + \frac{\theta^4}{\lfloor 4}$

$$e^{\theta} + e^{-\theta} = 2\left\{1 + \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} \cdots\right\}$$

$$e^{\theta} - e^{-\theta} = 2\left[\theta + \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} + \cdots\right]$$

here θ may be real or complex

Since $\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} = \frac{1}{\cancel{2}} \times \cancel{2}\left[1 + \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} + \cdots\right]$

$$= 1 + \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4}$$

$$= \sum_{n=0}^{\infty} \frac{\theta^{2n}}{\lfloor 2n}$$

Therefore, $\cosh\theta = \sum_{n=0}^{\infty} \frac{\theta^{2n}}{\lfloor 2n}$

<u>Similarly</u> $\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2} = \frac{1 \times \cancel{2}}{\cancel{2}}\left[\theta + \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} \cdots\right]$

$$= \theta + \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} \cdots$$

$$= \sum_{n=0}^{\infty} \frac{\theta^{2n+1}}{\lfloor 2n+1}$$

<u>Relation between the hyperbolic functions:</u> →

(1) (i) $\cosh^2\theta - \sinh^2\theta = 1$

(ii) $\operatorname{sech}^2\theta + \tanh^2\theta = 1$

(iii) $\coth^2\theta - \operatorname{cosech}^2\theta = 1$

(2) (i) $\sinh(\theta \pm \varphi) = \sinh\theta\cosh\varphi \pm \cosh\theta\sinh\varphi$

(ii) $\cosh(\theta \pm \varphi) = \cosh\theta\cosh\varphi \pm \sinh\theta\sinh\varphi$

(iii) $\tanh(\theta \pm \varphi) = \frac{\tanh\theta \pm \tanh\varphi}{1 \pm \tanh\theta\tanh\varphi}$

(3) (i)

$\sinh 2\theta = 2\sinh\theta\cosh\theta$

(ii) $\cosh 2\theta = \cosh^2\theta + \sinh^2\theta$ — $2\cosh^2\theta - 1$, $1 + 2\sinh^2\theta$.

(iii) $\tanh 2\theta = \dfrac{2\tanh\theta}{1+\tanh^2\theta}$.

Important deducted formulae

$$\cosh^2\theta = \frac{1+\cosh 2\theta}{2}$$

$$\sinh^2\theta = \frac{\cosh 2\theta - 1}{2}$$

* Expression of hyperbolic Functions in Terms of Circular Functions

Essential Requirement →

$$e^{i\theta} = \cos\theta + i\sin\theta.$$
$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

$i^{-1} = -i$, $i^2 = -1$
$i^{-2} = -1$, $i^3 = -i$
$i^{-3} = i$, $i^4 = 1$
$i^{-4} = 1$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

C.F → H.F

* $\sin i\theta = i\sinh\theta$

$\cos i\theta = \cosh\theta$

$\tan i\theta = i\tanh\theta$

H.F — C.F

$\sinh\theta = -i\sin i\theta$

$\cosh\theta = \cos i\theta$

$\tanh\theta = -i\tan i\theta$

Obviously $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$, $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

Replace θ by $i\theta$

$$\sin i\theta = \frac{e^{i^2\theta} - e^{-i^2\theta}}{2i}$$
$$= \frac{e^{-\theta} - e^{\theta}}{2i}$$
$$= -\frac{[e^{\theta} - e^{-\theta}]}{2} i^{-1}$$
$$= i\,\frac{e^{\theta} - e^{-\theta}}{2} = i\sinh\theta.$$

$$\cos i\theta = \frac{e^{i^2\theta} + e^{-i^2\theta}}{2}$$
$$= \frac{e^{-\theta} + e^{\theta}}{2}$$
$$= \frac{e^{\theta} + e^{-\theta}}{2}$$
$$= \cosh\theta.$$

Remarks: Hyperbolic Functions are wholly real

Nature of hyperbolic Functions

Since Necessary and sufficient Conditions under which

$f(\theta)$ is even

$$f(\theta) - f(\theta) = 0$$

Since $\cosh(-\theta) = \cosh\theta$

$\sinh(-\theta) = -\sinh\theta$

$\tanh(-\theta) = -\tanh\theta.$

is an odd function

$$f(\theta) + f(-\theta) = 0$$

clearly $f(\theta) = \cosh\theta$ is an even function.

$f(\theta) = \sinh\theta$ is an odd function.

$f(\theta) = \tanh\theta$ is an odd function.

Periods of the hyperbolic functions

Hyperbolic functions	Periods:
$f(\theta) = \cosh\theta$	$2\pi i$
$\sinh\theta$	$2\pi i$
$\tanh\theta$	πi

Remarks → Hyperbolic functions are periodic functions with imaginary period.